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A Representation Theorem for Circular Transforms

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INTRODUCTION

Let A be the class of functions defined and analytic on the annulus

$$S = \left\{ z \,\middle|\, \frac{1}{R} < |z| < R \right\},$$

where R is a fixed positive number. A circular transformation is a mapping K of A into itself given by

$$g(z) = \text{P.V.} \int_{|w|=1} k(wz) f(w)\, dw$$

which is such that $g = Kf$ implies $f = Kg$. The function k is called a "circular kernel". It was shown in [2] that k must have simple poles on the unit circle. It was also shown there that if k has the Laurent expansions

$$k(z) = \sum_{-\infty}^{\infty} \alpha_n z^n, \qquad 1 < |z| < R,$$

and

$$k(z) = \sum_{-\infty}^{\infty} \beta_n z^n, \qquad \frac{1}{R} < |z| < 1,$$

and

$$f(z) = \sum_{-\infty}^{\infty} a_n z^n, \qquad z \in S,$$

then

$$(Kf)(z) = \pi i \sum_{-\infty}^{\infty} (\alpha_n + \beta_n)\, a_{-1-n} z^n, \qquad z \in S.$$

The purpose of this paper is to obtain a representation theorem for Kf which isolates the roles that the poles and the analytic part of k play in this transformation.

With the introduction of the inner product

$$(f, g) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} \, d\theta,$$

the vector space A becomes a prehilbert space. The representation theorem that we obtain is used to show that any circular transformation is a bounded, linear transformation.

1. The Representation Theorem

We will need the following two lemmas, proven in [2].

Lemma 1. *A circular kernel k can be written*

$$k(z) = \sum_{i=1}^{m} \frac{\beta_i}{z - \alpha_i} + \sum_{-\infty}^{\infty} c_n z^n, \qquad R^{-1} < | z | < R,$$

where $\{\alpha_i\}$ are the poles of k on the unit circle and $\{\beta_i\}$ are the corresponding residues of k.

Lemma 2. *If h is analytic on S except for a simple pole at $z = e^{i\beta}$ with residue a there, and*

$$H(z) = h(z) - \frac{a}{z - e^{i\beta}},$$

then H is analytic on S and

$$\text{P.V.} \int_{|z|=1} h(z) \, dz = \pi a i + \int_{|z|=1} H(z) \, dz.$$

Theorem 1. *If f in A has the Laurent expansion*

$$f(z) = \sum_{-\infty}^{\infty} a_n z^n, \qquad R^{-1} < | z | < R, \tag{1}$$

then its circular transform, corresponding to the circular kernel k, has the representation

$$(Kf)(z) = \pi i \sum_{i=1}^{m} \frac{\beta_i}{z} f\left(\frac{\alpha_i}{z}\right) + \sum_{-\infty}^{\infty} c_n a_{-1-n} z^n \tag{2}$$

for z in S.

Proof. Using Lemma 1, we have

$$(Kf)(z) = \text{P.V.} \int_{|w|=1} k(zw) f(w)\, dw$$
$$= \text{P.V.} \int_{|w|=1} \left[\sum_{i=1}^{m} \frac{\beta_i}{wz - \alpha_i} + \sum_{-\infty}^{\infty} c_n z^n w^n \right] f(w)\, dw$$

for z on the unit circle. Since the series on the right is uniformly convergent, we have

$$(Kf)(z) = \sum_{i=1}^{m} \frac{\beta_i}{z} \text{P.V.} \int_{|w|=1} \frac{f(w)\, dw}{w - \alpha_i z^{-1}} + \sum_{-\infty}^{\infty} c_n z^n \int_{|w|=1} w^n f(w)\, dw,$$
$$| z | = 1.$$

If f has the Laurent expansion (1), the integral on the right above is the coefficient a_{-1-n} in that expansion. In order to demonstrate the validity of Eq. (2) on the unit circle, it remains to be shown that

$$\text{P.V.} \int_{|w|=1} \frac{f(w)\, dw}{w - \alpha_i z^{-1}} = \pi i f\left(\frac{\alpha_i}{z}\right), \qquad | z | = 1. \tag{3}$$

Using Lemma 2, we have

$$\text{P.V.} \int_{|w|=1} \frac{f(w)\, dw}{w - \alpha_i z^{-1}} = \pi i f\left(\frac{\alpha_i}{z}\right) + \int_{|w|=1} \frac{f(w) - f(\alpha_i z^{-1})}{w - \alpha_i z^{-1}}\, dw,$$

so that Eq. (3) reduces to

$$I = \int_{|w|=1} \frac{f(w) - f(\alpha_i z^{-1})}{w - \alpha_i z^{-1}}\, dw = 0.$$

We have

$$f(w) = \sum_{-\infty}^{\infty} a_n w^n, \qquad f(\alpha_i z^{-1}) = \sum_{-a}^{\infty} a_n \alpha_i^{\,n} z^{-n}$$

since the Laurent expansion of f converges in the annulus. Then

$$I = \int_{|w|=1} \sum_{-\infty}^{\infty} a_n \left(\frac{w^n - \alpha_i^{\,n} z^{-n}}{w - \alpha_i z^{-1}} \right) dw.$$

With the substitution

$$\xi = \frac{wz}{a_i},$$

we have

$$I = \int_{|w|=1} \sum_{-\infty}^{\infty} a_n \alpha_i^n z^{-n} \left(\frac{\xi^n - 1}{\xi - 1}\right) d\xi$$

$$= \int_{|w|=1} \left[\sum_{-\infty}^{\infty} a_n \alpha_i^n z^{-n} (\xi^{n-1} + \xi^{n-2} + \cdots + \xi + 1)\right] d\xi.$$

Proceeding formally, we have

$$I = \sum_{-\infty}^{\infty} a_n \alpha_i^n z^{-n} \int_{|\xi|=1} (\xi^{n-1} + \xi^{n-2} + \cdots + \xi + 1)\, d\xi = 0.$$

This formal interchange of summation and integration is justified if the series converges uniformly. We demonstrate this using the Weierstrass test for uniform convergence of a series of functions. We have

$$| a_n \alpha_i^n z^{-n} (\xi^{n-1} + \cdots + 1)| \leqslant | n - 1 | \, | a_n | < | n | \, | a_n |$$

since

$$| \alpha_i^n z^{-n} | = 1,$$

and the series

$$\sum_{-\infty}^{\infty} | n | \, | a_n |$$

is convergent because $f'(z)$ is analytic in the annulus and its Laurent series is absolutely convergent there.

We have demonstrated the validity of Eq. (2) for z on the unit circle. Let g be defined on the unit circle by

$$g(z) = \pi i \sum_{i=1}^{m} \frac{\beta_i}{z} f\left(\frac{\alpha_i}{z}\right) + \sum_{-\infty}^{\infty} c_n a_{-1-n} z^n.$$

This formula actually defines an analytic function in S, since

$$\frac{1}{z} f\left(\frac{1}{z}\right) = \sum_{-\infty}^{\infty} a_{-1-n} z^n$$

is analytic in S and $\{c_n\}$ is a bounded sequence. Then Kf and g define the same function on S since they agree on the unit circle.

Remark 1. The proof of Theorem 1 shows that

$$(Kf)(z) = \pi i \sum_{i=1}^{m} \frac{\beta_i}{z} f\left(\frac{\alpha_i}{z}\right) + \int_{|w|=1} k_0(zw) f(w)\, dw \tag{4}$$

where k_0, given by

$$k_0(z) = \sum_{-\infty}^{\infty} c_n z^n, \qquad z \in S,$$

is the analytic part of k in S. This shows that K is a finite sum of bounded operators mapping the prehilbert space A into itself, and, hence

$$K : A \to A$$

is a bounded linear operator.

Remark 2. In case the analytic part of k vanishes, then

$$(Kf)(z) = \pi i \sum_{i=1}^{m} \frac{\beta_i}{z} f\left(\frac{\alpha_i}{z}\right).$$

Circular kernels of this type were called "Finite Reciprocities" in [1].

References

1. L. Artiaga, Finite reciprocities, *Canad. Math. Bull.* 1 (1964), 283–290.
2. L. Artiaga, Circular transformations, *J. Math. Anal. Appl.* 13 (1966), 475–485.